Helicity-type integral invariants for Hamiltonian systems

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Abstract

In this note, we consider generalizations of the asymptotic Hopf invariant, or helicity, for Hamiltonian systems with one-and-a-half degrees of freedom and symplectic diffeomorphisms of a two-disk to itself.

1   Helicity integral for Hamiltonian systems

Consider a Hamiltonian system on the extended phase space $\mathbb{R}^2 \times S^1$ with symplectic coordinates $p, q \in \mathbb{R}^2$ and time $S^1 = \{t \pmod{2\pi}\}$, given by a Hamiltonian $H(p, q, t)$:

$$\dot{p} = -H_q, \quad \dot{q} = H_p, \quad \dot{t} = 1. \tag{1.1}$$

Let $D \in \mathbb{R}^2$ be an invariant disk for this system (which always exists for integrable systems with bounded energy levels and near-integrable systems, close to them). An integral over a solid torus $D \times S^1$:

$$\mathcal{H}(H) = \int_0^{2\pi} \int_D \tilde{H}(p, q, t) \, dp \wedge dq \wedge dt, \tag{1.2}$$

where the function $\tilde{H} = H + const$ equals zero on the boundary $\mathbb{T}^2 = \partial D \times S^1$, will be called *helicity* for the Hamiltonian system (1.1). It is easy to see that this definition is consistent – the value $\mathcal{H}$ does

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not depend on the choice of the symplectic coordinates. The term "helicity" is chosen, as the integral (1.2) equals (up to a multiplier 2) to the helicity of the Hamiltonian vector field $\xi = (-H_q, H_p, 1)$ (with invariant measure on the extended phase space being $\mu = dp \wedge dq \wedge dt$):

$$2 \int_{D \times S^1} \tilde{H}(p, q, t) dp \wedge dq \wedge dt = \int_{D \times S^1} i_\xi \mu \wedge (i_\xi \mu)^{-1}, \qquad (1.3)$$

provided the 1-form $(i_\xi \mu)^{-1}$ is chosen such that

$$\int_{t \in S^1} (i_\xi \mu)^{-1} = 0. \qquad (1.4)$$

Indeed, as $i_\xi \mu = dp \wedge dq - dH \wedge dt$, then, due to (1.4), $(i_\xi \mu)^{-1} = p dq - \tilde{H} dt$. Relation (1.3) is now obtained by the integration by parts, cf. [10]. Notice that, as we have assumed that the disk D is invariant, from Condition (1.4) follows that $\tilde{H} = 0$ on its boundary ∂D.

The helicity invariant has the following ergodic interpretation: it measures asymptotic linking of the trajectories of the Hamiltonian vector field $(-H_q, H_p, 1)$ in the solid torus $D \times S^1$, which in turn is itself unlinked (and untwisted), see [10] for details. In particular, the invariant (1.2) does not change after rescaling $H \to \mu H$, $t \to \frac{1}{\mu} t$ – the average linking of the trajectories remain the same.

Remark. Expression (1.2) is still well defined if the Hamiltonian H is discontinuous at some t. One can show that the ergodic interpretation of (1.2) remains the same.

Expression (1.2) is exactly the same as the expression for the Calabi invariant, see [8]. The difference is that above we have not assumed that the gradient of the Hamiltonian H is zero on the boundary torus $\mathbb{T}^2$ (which is an assumption in the Calabi invariant definition). Of course if we have assumed that, then (1.2) would be the Calabi invariant.

2   Hamiltonian diffeomorphisms

Consider a symplectic mapping $h : D \to D$. There exists a Hamiltonian system (with the Hamiltonian H depending 2π-periodically on time), which Poincaré map $g_H^{2\pi}$ coincides with h (actually, there are infinitely many such systems, and one can show that they have the same smoothness as the mapping itself, see, e.g., [18]). Notice that for the calculations below, it does not matter which period to take (due to the ergodic interpretation of the helicity integral).

The Calabi invariant is an invariant of the symplectic mapping, identical at the boundary, and it does not depend on a paricular choice of the underlying Hamiltonian system, see, e.g., [3]. On the contrary, if we only demand that the mapping h sends the boundary ∂D to itself, then the value of the integral (1.2) may depend on a particular choice of the Hamiltonian flow. However, the values of $\mathcal{H}$ will be "quantized" by the square of the sympectic area $S(D)$ of the disk D:

Theorem 2.1. *Let a symplectic mapping $h : D \to D$ be given, which sends a boundary ∂D to itself. Let $g_{H_1}^t$ and $g_{H_2}^t$ be two Hamiltonian flows, given by 2π-periodic Hamiltonians H_1 and H_2, such that $g_{H_1}^{2\pi} = g_{H_2}^{2\pi} = h$. Then, $\mathcal{H}(H_1) - \mathcal{H}(H_2) = nS(D)^2/2$ for some $n \in \mathbb{Z}$.*

Proof. We first prove the following

Lemma 2.2. *Let $h = id$. Then for any H, such that $g_H^{4\pi} = id$, $\mathcal{H}(H) = nS(D)^2/2$, $n \in \mathbb{Z}$.*

Proof of Lemma 2.2. The mapping $g_H^{4\pi}$ turns the boundary circle by the angle $2\pi n$, $n \in \mathbb{Z}$ (because of continuity, no other transformation of the circle than the pure rotation is allowed). Take a bigger disk $\tilde{D}$, which contains D, and define a mapping $\tilde{h} : \tilde{D} \to \tilde{D}$, stationary at the boundary $\partial \tilde{D}$, and coinciding with h in D. The Calabi invariant for the mapping $\tilde{h}$ does not depend on an underlying Hamiltonian flow. If we take the flow that coincides with g_H^t inside D, and that is stationary at the outer boundary $\partial \tilde{D}$, then, as we tend $\tilde{D} \to D$, its Calabi invariant will tend to $\mathcal{H}(H)$, as $H|_{\partial D}$ will tend to zero as $\tilde{D} \to D$.

Now, we take the Hamiltonian $\hat{H}$ in $\tilde{D}$, such that in D, $\hat{H} = nI/2 + c$, where I, ϕ are the "action-angle" variables, such that the disk D is given by inequality $I \leq I_0$, and the disk $\tilde{D}$ is given by $I \leq I_1$, $I_1 > I_0$. We define the constant c later. Let $I_1 = I_0 + \epsilon$. The function $\hat{H}$ in $\tilde{D} \setminus D$ can be taken as

$$\hat{H} = \frac{n}{4\epsilon}(-I + I_0 + \epsilon)^2. \qquad (2.1)$$

Indeed, $\hat{H}|_{\partial \tilde{D}} = 0$, $d\hat{H}|_{\partial \tilde{D}} = 0$ and

$$\frac{d\hat{H}}{dI}(I_0) = \frac{n}{2\epsilon}(-I_0 + I_0 + \epsilon) = \frac{n}{2}.$$

As $\hat{H}(I_0) = n\epsilon/4$, the constant $c = n\epsilon/4 - nI_0/2$.

The Calabi invariant equals

$$\mathcal{C}(\tilde{h}) = 4\pi \int_{\tilde{D}} \hat{H} dI \wedge d\phi = 4\pi^2 n \int_0^{I_0} (I - I_0) dI + O(\epsilon).$$

As we turn $\epsilon \to 0$, we get

$$\mathcal{C}(\tilde{h}) \to -2\pi^2 n I_0^2 = -\frac{nS(D)^2}{2},$$

where $S(D)$ is the symplectic area of the disk D. $\square$

Now, to prove the theorem, consider a mapping $g_{H_1}^{2\pi} \cdot g_{-H_2}^{2\pi}$. This mapping is identical. Its asymptotic Hopf invariant equals sum of asymptotic Hopf invariants for systems with Hamiltonians H_1 and $-H_2$, which, by Lemma 2.2, equals $nS(D)^2/2$. $\square$

Remark. The group of Hamiltonian diffeomorphisms of a 2-disk, identical at the boundary, is known to be contractible, see [11]. In particular, the Calabi 1-form, defined as the right-invariant differential form, coinciding with the Calabi integral

$$\int_D H \, dp \wedge dq$$

on the Lie algebra of Hamiltonian diffeomorphisms of disk D, is exact, see [3]. If we drop the condition that a Hamiltonian diffeomorphism is identical at the boundary, then from Theorem 2.1 follows that this form (which is still correctly defined and closed) is no longer exact, and thus the topology of the group of Hamiltonian diffeomorphisms becomes more complicated.

The statement converse to Theorem 2.1 is also true. Let a smooth symplectic mapping $h : D \to D$ and a number $n \in \mathbb{Z}$ be given.

Theorem 2.3. *There are two smooth Hamiltonian systems, given by Hamiltonians H_1 and H_2, 2π-periodic in t, such that $h = g_{H_1}^{2\pi} = g_{H_2}^{2\pi}$, and*

$$\mathcal{H}(H_1) - \mathcal{H}(H_2) = nS(D)^2/2.$$

Proof. It is well-known that a symplectic mapping can be inserted into a Hamiltonian flow, see, e.g., [18]. Let this flow be given by a Hamiltonian H_1 (which is zero on the boundary ∂D). Let I, ϕ be the "action-angle" variables, such that the disk D is given by $I = I_0$. Take now the following function:

$$\tilde{H}_2(I, \phi, t) = \left\{ \begin{array}{ll} \frac{1}{2} H_1(I, \phi, 2t), & t \in [0, \pi] \\ 2n(I - I_0), & t \in [\pi, 2\pi]. \end{array} \right.$$

Obviously, $\mathcal{H}(\tilde{H}_2) - \mathcal{H}(H_1) = \frac{n}{2} S(D)^2$. Now, we have to construct a smooth Hamiltonian H_2 with the same helicity as $\tilde{H}_2$. We show that this can be done by time transformation.

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Suppose that we want to make H_2 differentiable (once, in time). Consider the following time transformation: $t = \tau - \frac{1}{2}\sin 2(\tau - \pi)$. The helicity integral becomes:

$$\mathcal{H}(\tilde{H}_2) = \int_0^{2\pi} \int_D (1 - \cos 2(\tau - \pi))\tilde{H}_2(I, \phi, \tau)dI \wedge d\phi \wedge d\tau =$$

$$= \int_0^{2\pi} \int_D H_2 \, dI \wedge d\phi \wedge d\tau = \mathcal{H}(H_2),$$

where we denoted $H_2(I, \phi, \tau) = (1 - \cos 2(\tau - \pi))\tilde{H}_2(I, \phi, \tau)$. Obviously, $H_2 \in C^1$ (being a 2π-periodic function of τ).

To obtain smoothness of order $k \in \mathbb{N}$, one can construct time transformation, which is given by $t = \tau + (\text{periodic function of } \tau)$, such that at $\tau = \delta$ and $\tau = \pi + \delta$, $t = O(\delta^{k+2})$ and $t = \pi + O(\delta^{k+2})$ correspondingly for small δ. For example, for $k = 3$, take

$$t = \tau - \frac{1}{2}\sin 2(\tau - \pi) - \frac{1}{12}\sin^3 2(\tau - \pi). \qquad \square$$

Remark. Theorem 2.3 is also true in the analytic case. The proof is simple modulus results in [18]: one should for example use the smoothing technique from [18] to get an analytic system from the smooth one, obtained during the proof of Theorem 2.3. Notice also that if two Hamiltonian functions are close to each other (in an appropriate topology) and they define the same mapping, then the helicity will also be the same – by continuity of the helicity functional and by Theorem 2.1.

From Theorem 2.3 follows that a symplectic mapping can be inserted into a Hamiltonian flow with any given helicity level. In particular, there exists the flow with smallest (in the absolute value) helicity, which is a functional of the mapping only. This smallest helicity will be called the *generalized Calabi invariant*.

We do not write down explicitly the generalized Calabi invariant as the functional of a symplectomorphism here. We only note that it can be written as a sum of two parts: an integral over the disk, and an integral over the boundary circle. One way to see that is to take the limiting procedure, as we did in the proof of Lemma 2.2.

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